Proof of a Theorem of Tchebycheff's on Definite Integrals.

By F. Franklin.

M. Hermite, in his *Cours professé pendant le* 2^e *Semestre*, 1881–82, states the following theorem as communicated to him by M. Tchebycheff:

Let u and v be two functions of x which between the values $x = 0$ and $x = 1$ are positive, and both vary in the same sense, so that they shall be continually increasing or continually decreasing; then we shall have the inequality

$$\int_0^1 uvdx > \int_0^1 udx \int_0^1 vdx.$$

But, supposing that one of the functions be increasing and the other decreasing, we shall have, on the contrary,

$$\int_0^1 uvdx < \int_0^1 udx \int_0^1 vdx.$$

M. Hermite proceeds to give a proof of this theorem by M. Picard, which is somewhat indirect and long. The theorem may be proved instantaneously, and no restriction need be placed on the signs of u and v; it admits, too, of an extension to the case of any real limits. The theorem, as thus enlarged, is as follows:

If u and v be two functions of x, both increasing continually or both decreasing continually as x passes from a to b, then

$$(b-a)\int_a^b uvdx > \int_a^b udx \int_a^b vdx;$$

and if one increases throughout while the other decreases throughout,

$$(b-a)\int_a^b uvdx < \int_a^b udx \int_a^b vdx.$$

The proof is as follows. Write $u = f(x)$, $v = \phi(x)$; then

$$\begin{aligned}\int_a^b \int_a^b [f(x)-f(y)][\phi(x)-\phi(y)]\,dxdy \\ &= (b-a)\int_a^b f(x)\,\phi(x)\,dx + (b-a)\int_a^b f(y)\,\phi(y)\,dy \\ &\quad - \int_a^b f(x)\,dx \int_a^b \phi(y)\,dy - \int_a^b \phi(x)\,dx \int_a^b f(y)\,dy \\ &= 2\left\{(b-a)\int_a^b uvdx - \int_a^b udx \int_a^b vdx\right\}.\end{aligned}$$

Now if u and v are always increasing or always decreasing, $f(x)-f(y)$ and $\phi(x)-\phi(y)$ always have like signs, while if one is increasing and the other decreasing they have opposite signs; hence in the former case the double integral above written is necessarily positive, while in the latter case it is necessarily negative. This proves the theorem.

In the same way we may prove the theorem about finite series from which the preceding may at once be deduced, viz.

$$n\Sigma_1^n u_r v_r > \Sigma_1^n u_r \Sigma_1^n v_r \quad \text{or} \quad n\Sigma_1^n u_r v_r < \Sigma_1^n u_r \Sigma_1^n v_r,$$

according as the u's and v's are both ascending (or both descending) series, or one of them ascending and the other descending.

The above demonstration was suggested by the following very simple method of proving either theorem, which I shall give only as applied to the second; the proof for the first being precisely similar. Consider the rectangular matrices

$$\begin{vmatrix} u_1, & u_2, & u_3, & \ldots & u_n \\ 1, & 1, & 1, & \ldots & 1 \end{vmatrix}, \quad \begin{vmatrix} v_1, & v_2, & v_3, & \ldots & v_n \\ 1, & 1, & 1, & \ldots & 1 \end{vmatrix}.$$

This product is the determinant

$$\begin{vmatrix} \Sigma uv, & \Sigma u \\ \Sigma v, & n \end{vmatrix}, = n\Sigma uv - \Sigma u \Sigma v.$$

But the product of the matrices is equivalent to the sum of the products obtained by multiplying every determinant in the first matrix by the corresponding one in the second, and any such product is $(u_r - u_s)(v_r - v_s)$, which is necessarily positive if the series both ascend or both descend throughout, and necessarily negative if one ascends throughout while the other descends throughout.

It is obvious from the nature of the proof that we may give the theorems a somewhat greater generality, viz., they hold whenever u and v vary throughout the interval in such a way that the greater u is the greater v is, or that the greater u is the smaller v is.* As examples of this somewhat more general theorem, we may note that the inequalities

$$\left(\int_a^b u\,dx\right)^2 < (b-a)\int_a^b u^2 dx, \quad \int_a^b u\,dx \int_a^b \frac{dx}{u} > (b-a)^2,$$

hold without any restriction as regards the manner in which u varies.

I add a few particular examples of the theorems, which may not be without interest.

* This, it should be noted, is not the same thing as saying that u and v increase together (or the opposite) as x passes from a to b; the theorem does not necessarily hold when this latter condition is fulfilled.

1. $x\int_0^x \frac{dx}{\sqrt{1-x^2}} < \int_0^x \frac{dx}{\sqrt{1-x}} \int_0^x \frac{dx}{\sqrt{1+x}}$; $\sin^{-1}x < \frac{4}{x}(\sqrt{1+x}+\sqrt{1-x}-\sqrt{1-x^2}-1)$;

$\frac{\pi}{2} < 4(\sqrt{2}-1)$, $\frac{\pi}{12} < 2(\sqrt{6}+\sqrt{2}-\sqrt{3}-2)$.

2. $\left(\int_0^x \frac{dx}{\sqrt{1-x^2}}\right)^2 < x\int_0^x \frac{dx}{1-x^2}$; $(\sin^{-1}x)^2 < \frac{x}{2}\log\frac{1+x}{1-x}$; $\frac{\pi^2}{36} < \frac{1}{4}\log 3$.

3. $\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \sin^2 x dx > \left(\int_0^{\frac{\pi}{2}} \sin x dx\right)^2$; $\frac{\pi^2}{8} > 1$.

4. $\int_1^x \frac{dx}{x}\int_1^x x dx > (x-1)^2$; $\log x > 2\frac{x-1}{x+1}\ (x>1)$; $\log x < 2\frac{x-1}{x+1}\ (x<1)$.

$\left(\int_1^x \frac{dx}{x}\right)^2 < (x-1)\int_1^x \frac{dx}{x^2}$; $\log x < \frac{x-1}{\sqrt{x}}$, $(x>1)$; $\log x < \frac{x-1}{\sqrt{x}}\ (x<1)$.

Thus $\frac{1}{\sqrt{2}} > \log 2 > \frac{2}{3}$.

5. $\int_0^1 x^{a-1}(1-x)^{b-1}dx < \int_0^1 x^{a-1}dx \int_0^1 (1-x)^{b-1}dx$; $ab\Gamma(a)\Gamma(b) < \Gamma(a+b)$, or $\Gamma(a+1)\Gamma(b+1) < \Gamma(a+b)$.

In particular (a being supposed between 0 and 1) $\Gamma(a)\Gamma(1-a) < \frac{1}{a(1-a)}$, so that $\frac{\pi}{\sin a\pi} < \frac{1}{a(1-a)}$, whence $\sin x > x\left(1-\frac{x}{\pi}\right)$.

6. $K\int_0^K \mathrm{sn}^2 u du > \left(\int_0^K \mathrm{sn}\, u du\right)^2$; $K(K-E) > \left(\log\frac{k'}{1-k}\right)^2$

$K\int_0^K \mathrm{sn}^2 u du < \int_0^K (1-\mathrm{cn}\, u)du \int_0^K (1+\mathrm{cn}\, u)du$; $K(K-E) < k^2K^2 - \left(\tan^{-1}\frac{k}{k'}\right)^2$.

The comparison of these inequalities gives superior and inferior limits for E, viz.

$$K - \frac{1}{K}\left(\log\frac{k'}{1-k}\right)^2 > E > k'^2K + \frac{1}{K}\left(\tan^{-1}\frac{k}{k'}\right)^2.$$

7. $x\int_0^x \frac{dx}{\sqrt{1-x^2.1-k^2x^2}} < \int_0^x \frac{dx}{\sqrt{1-x.1-kx}} \int_0^x \frac{dx}{\sqrt{1+x.1+kx}}$

8. $x\int_0^x \frac{dx}{\sqrt{1-x^2.1-k^2x^2}} > \int_0^x \frac{dx}{\sqrt{1-x^2}} \int_0^x \frac{dx}{\sqrt{1-k^2x^2}}$; $\int_0^x \frac{dx}{\sqrt{1-x^2.1-k^2x^2}} > \sin^{-1}x \frac{\sin^{-1}kx}{kx}$,

or $$F(k, \phi) > \frac{\varphi \sin^{-1}(k\sin\varphi)}{k\sin\varphi};\ K > \frac{\pi}{2}\frac{\sin^{-1}k}{k}.$$

9. $\int_0^\phi \sqrt{1-k^2\sin^2\varphi}\,d\varphi \int_0^\phi \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}} > \phi^2$; $E(k,\phi)F(k,\phi) > \phi^2$; $EK > \frac{\pi^2}{4}$.

Finally we may notice an obvious geometrical interpretation of the inequality

$$\int_a^b u dx \int_a^b \frac{dx}{u} > (b-a)^2;$$

viz., if the inverse of a closed curve be taken with respect to a circle whose centre is a point within the curve (and such that chords through it cut the curve in only two points), the area of the circle is always less than a geometric mean between the areas of the two curves.